Homotopy Groups in O-minimal Structures

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General Homotopy Groups

Let X, Y be topological spaces and $f, g : X \to Y$ (continuous) functions. We say that f and g are homotopic, denoted $f \simeq g$, if there exists a function

$$F : X \times [0,1] \to Y$$

such that, for all $x \in X$, we have $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

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For $x_0 \in X$ and $n \in \mathbb{N}$, the set of homotopy classes $[f]$ of maps

$$f : ([0,1]^n, \partial[0,1]^n) \to (X, x_0)$$

are the elements of the n-th homotopy group $\pi_n(X, x_0)$.

General Homotopy Groups

For each map $h : (X, x_0) \to (Y, y_0)$ and $n \in \mathbb{N}$ we define

$$h_* : \pi_n(X, x_0) \to \pi_n(Y, y_0) : [f] \to [h \circ f].$$

We call h_* the induced homomorphism.

We say that topological spaces X and Y are homotopy equivalent if there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq \mathbb{1} \text{ and } f \circ g \simeq \mathbb{1}.$$

In this case, then the induced homomorphisms f_* and g_* are isomorphisms.

The O-minimal Field Case

Fix an o-minimal expansion $\mathcal{R}$ of a real closed field R and let $\mathcal{R}_0$ be the field structure.

Let (X, x_0) be a definable pointed set and $n \in \mathbb{N}$. We define the o-minimal homotopy group $\pi_n(X, x_0)^{\mathcal{R}}$ in the obvious way.

Theorem (Triangulation theorem)

Every definable set $X \subseteq R^m$ is definably homeomorphic to a definable simplicial complex S.

Theorem (Baro, Otero, 2008)

For every semialgebraic pointed set (X, x_0) and $n \in \mathbb{N}$, the map $\rho : \pi_n(X, x_0)^{\mathcal{R}_0} \to \pi_n(X, x_0)^{\mathcal{R}} : [f] \to [f]$ is an isomorphism.

The O-minimal Group Case

Fix an o-minimal expansion $\mathcal{G}$ of an ordered group G and let $\mathcal{G}_0$ be the group structure.

Issue: Not all closed intervals are in definable bijection with each other.

Let $f, g : X \to Y$ be definable functions. Then, f and g are definably homotopic if, for some $q \in G$ with $q > 0$, there exists a definable function

$$F : X \times [0, q] \to Y$$

such that, for all $x \in X$, we have $F(x, 0) = f(x)$ and $F(x, q) = g(x)$.

For $x_0 \in X$, $n \in \mathbb{N}$ and $p \in G$ with $p > 0$, the set of definable homotopy classes $[f]$ of definable maps

$$f : ([0, p]^n, \partial[0, p]^n) \to (X, x_0)$$

are the elements of the n-th o-minimal homotopy group $\pi_{(n,p)}(X, x_0)$ of size p.

The O-minimal Group Case

For $p, q \in G$ with $p > q > 0$, we define

$$h^1_{(p,q)} : [0,p] \to [0,q] : x \mapsto \begin{cases} x & \text{for } 0 \leq x < \frac{q}{2}, \\ \frac{q}{2} & \text{for } \frac{q}{2} \leq x < p - \frac{q}{2}, \\ x - (p-q) & \text{for } p - \frac{q}{2} \leq x \leq p, \end{cases}$$

and for each $n \in \mathbb{N}$ we define

$$h^n_{(p,q)} : [0,p]^n \to [0,q]^n : (x_1, \ldots, x_n) \mapsto (h^1_{(p,q)}(x_1), \ldots, h^1_{(p,q)}(x_n)).$$

This gives us a homomorphism

$$\overline{h^n_{(p,q)}} : \pi_{(n,q)}(X, x_0)^{\mathcal{G}} \to \pi_{(n,p)}(X, x_0)^{\mathcal{G}} : [f] \mapsto [f \circ h].$$

The O-minimal Group Case

Theorem

Let (X, x_0) be a definable pointed set with X bounded and let d be the diameter of X. Then, for each $n \in \mathbb{N}$ and all $p \in G$ with $p > d$, we have $\pi_{(n,d)}(X, x_0)^{\mathcal{G}} \cong \pi_{(n,p)}(X, x_0)^{\mathcal{G}}$.

Theorem (?)

Let (X, x_0) be a definable pointed set. Then, there exists a definable bounded subset $X' \subseteq X$ with $x_0 \in X'$ such that, for all $n \in \mathbb{N}$ and $p \in G$ with $p > 0$, we have $\pi_{(n,p)}(X', x_0)^{\mathcal{G}} \cong \pi_{(n,p)}(X, x_0)^{\mathcal{G}}$

Let (X, x_0) be a definable pointed set with X bounded and $n \in \mathbb{N}$. We define the n-th o-minimal homotopy group

$$\pi_n(X, x_0)^{\mathcal{G}} := \pi_{(n,d)}(X, x_0)^{\mathcal{G}}$$

with d the diameter of X.

The O-minimal Group Case

Issue: The classical definition of homotopy equivalence seems insufficient here.

Topological spaces X and Y are weakly homotopy equivalent if there exist a map $f : X \to Y$ such that the induced homomorphism f_* are isomorphisms.

Theorem (Whitehead's Theorem)

Two CW complexes X and Y are weakly homotopy equivalent if and only if they are homotopy equivalent.

Theorem (Baro, Otero, 2008)

In the o-minimal field case, two definable sets X and Y are weakly homotopy equivalent if and only if they are homotopy equivalent.

In the o-minimal group case, the homotopy equivalence relationship is defined as the symmetric closure of the weakly homotopy equivalence relationship.

The O-minimal Group Case

We do not have triangulation in the traditional sense.

Issue: Not all definable sets are triangulable.

Issue: For $k \in \mathbb{N}$, not all k-simplices are in definable bijection.

We do have the following theorem.

Theorem (H-triangulation Theorem)

For every $\mathcal{G}$-definable set X, there exists a $\mathcal{G}_0$-definable normalised simplicial complex S such that X and $|S|$ are homotopy equivalent.

Theorem

Let S be a $\mathcal{G}_0$-definable normalised simplicial complex, $x_0 \in S$ and $n \in \mathbb{N}$. Then,

$$\pi_n(S, x_0)^{\mathcal{G}} \cong \pi_n(S, x_0)^{\mathcal{G}_0}.$$

Thank you for your attention!